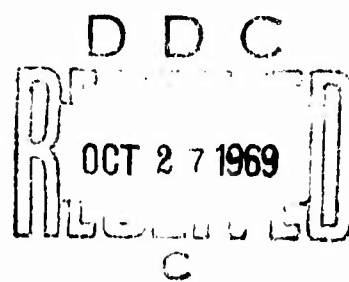AD695444

MEMORANDUM
RM-6102-PR
SEPTEMBER 1969

DISJOINT COMMON
PARTIAL TRANSVERSALS OF
TWO FAMILIES OF SETS

D. R. Fulkerson

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The RAND *Corporation*

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PREFACE

The theory of flows in capacity-constrained networks has been a useful tool in the study of a variety of combinatorial problems. Another application of this kind is discussed in this Memorandum, which continues RAND's basic work on topics in combinatorial mathematics. Earlier relevant RAND publications include:

> R-375-PR, <u>Flows in Networks</u>, L. R. Ford, Jr., and D. R. Fulkerson, August 1962.

> RM-1977-PR, <u>Network Flow and Systems of Representatives</u>, L. R. Ford, Jr., and D. R. Fulkerson, September 1957.

> RM-5061-PR, <u>Edge Colorings in Bipartite Graphs</u>, J. H. Folkman, and D. R. Fulkerson, August 1966.

> RM-5834-PR, <u>Blocking Polyhedra</u>, D. R. Fulkerson, December 1968.

The specific problem treated in this Memorandum is that of finding necessary and sufficient conditions in order that two finite families of subsets of a finite set possess k mutually disjoint common partial transversals, each of prescribed size p. A mor general problem, mentioned but not solved in the paper, is that of characterizing the blocking matrix of the incidence matrix of all common partial transversals, of size p, of two finite families of subsets of a finite set.

SUMMARY

The theory of flows in networks is applied to obtain
necessary and sufficient conditions on two finite families
of subsets of a finite set in order that there exist k
mutually disjoint common partial transversals, each of
size p, of the two families.

CONTENTS

DISJOINT COMMON PARTIAL TRANSVERSALS OF
TWO FAMILIES OF SETS

1. <u>Introduction</u>. Let $\mathcal{A} = (A_i : i \epsilon I)$ and $\mathcal{B} = (B_j : j \epsilon J)$ be
two families of subsets of a finite set E, the cardinalities
$|I|$ and $|J|$ of the index sets I and J being assumed finite
also. A subset P of E, with $|P| = p$, is a <u>partial</u>
<u>transversal</u> (of size p) of the family $\mathcal{A}$ if there is a
one-one mapping φ from P into I such that $e \epsilon A_{\varphi(e)}$ for
all $e \epsilon P$. If P is a partial transversal of $\mathcal{A}$ and also of
$\mathcal{B}$, then P is a <u>common partial transversal</u> of $\mathcal{A}$ and $\mathcal{B}$. A
partial transversal of $\mathcal{A}$ of size $p = |I|$ is a <u>transversal</u>
of $\mathcal{A}$, and, if $|I| = |J|$, a common partial transversal of
$\mathcal{A}$ and $\mathcal{B}$ of size $p = |I|$ is a <u>common transversal</u> of $\mathcal{A}$ and $\mathcal{B}$.

In this note we show how the theory of flows in networks
[3] can be used to solve a packing problem for common
partial transversals, of prescribed size p, of the families
$\mathcal{A}$ and $\mathcal{B}$. Specifically, we answer the question: What are
necessary and sufficient conditions in order that there
exist k mutually disjoint common partial transversals,
each of size p, of the two families $\mathcal{A}$ and $\mathcal{B}$?

2. <u>A packing theorem</u>. For the case $k = 1$ and $p = |I| = |J|$,
the following theorem is known [3,4].

Theorem 2.1. The finite families $\mathcal{A} = (A_i : i \in I)$ and $\mathcal{B} = (B_j : j \in J)$ of subsets of the finite set E, with $|I| = |J|$, have a common transversal if and only if

(2.1) $|I'| + |J'| - |I| \leq | \underset{i \in I'}{\cup} A_i \cap \underset{j \in J'}{\cup} B_j |$

holds for all $I' \subseteq I$, $J' \subseteq J$.

The inequalities (2.1) can be generalized in a simple way to provide an answer to the question posed above. This generalization is described below in Theorem 2.2 (ii); an equivalent set of conditions is provided by (iii) of Theorem 2.2. To state (iii), we use the notation, for $E' \subseteq E$:

(2.2) $\#(\mathcal{A}, E') = |\{i \in I / e \in A_i \text{ for some } e \in E'\}|.$

In words, $\#(\mathcal{A}, E')$ is the number of members of the family $\mathcal{A}$ represented by elements of E'.

Theorem 2.2. Let $\mathcal{A} = (A_i : i \in I)$ and $\mathcal{B} = (B_j : j \in J)$ be finite families of subsets of the finite set E. Then the following statements are equivalent.
(i) There exist k mutually disjoint common partial transversals, each of size p, of $\mathcal{A}$ and $\mathcal{B}$.
(ii) The inequality

(2.3) $k[p-(|I-I'| + |J-J'|)] \leq | \underset{i \in I'}{\cup} A_i \cap \underset{j \in J'}{\cup} B_j |$

holds for all $I' \subseteq I$, $J' \subseteq J$.

(iii) <u>The inequality</u>

(2.4) $k[p-(\#(\mathcal{A},E') + \#(\mathcal{B},E''))] \le |E-(E'\cup E'')|$

<u>holds</u> <u>for</u> <u>all</u> $E' \subseteq E$, $E'' \subseteq E$.

We defer the proof of Theorem 2.2 to Section 4, following the discussion in Section 3 of a certain decomposition theorem for network flows that will be needed in the proof. Here we note only that it is enough to state (2.4) for all disjoint subsets E', E'' of E, since this yields an equivalent set of conditions.

3. <u>Flows and matching flows.</u> Let f be an integral flow from a set S of sources to a set T of sinks in a (directed) network having node set N. Thus S and T are disjoint subsets of N, and

(3.1) $f(x,N) - f(N,x) = \begin{cases} a(x), x \in S, \\ -b(x), x \in T, \\ 0, x \in N-(S\cup T), \end{cases}$

(3.2) $f(x,y) \ge 0, (x,y) \in N \times N.$

Here $a(x)$, $x \in S$, and $b(x)$, $x \in T$, are nonnegative integers, and $v = a(S) = b(T)$ is the size (or amount) of the flow f. (We are using the notation of $|3|$, e.g. $f(x, N) = \sum_{y \in N} f(x,y)$, and $a(S) = \sum_{x \in S} a(x)$.)

If $a(x) = 0$ or 1 for all $x \in S$, and $b(x) = 0$ or 1 for all $x \in T$, we say that f is a <u>matching flow</u> from S to T.

An integral flow from S to T of amount v can be decomposed into a sum of an integral cyclic flow, or circulation, (all right-hand sides zero in (3.1)), and v unit chain-flows, where each chain begins at a node of S and ends in a node of T [3]. In particular, if f is a matching flow, the chain-flows in such a decomposition induce a matching between those nodes $x \in S$ such that $a(x) = 1$ and those nodes $x \in T$ such that $b(x) = 1$, by pairing first and last members of each chain. (Several different matchings can result in this way, since the decomposition of f need not be unique.) This justifies calling f a matching flow.

For the proof of Theorem 2.2, we shall be interested in decomposing an arbitrary integral flow from S to T into a sum of matching flows, each of the same size.

Theorem 3.1. Let f be an integral flow from S to T of size $v = a(S) = b(T)$, and let

$$(3.3) \qquad m = \max \left(\max_{x \in S} a(x), \max_{x \in T} b(x) \right).$$

Then f decomposes into a sum of k matching flows, each of size p, if and only if $v = kp$ and $m \leq k$.

Proof. Necessity is clear. To prove sufficiency, we can proceed as follows. Decompose f into the sum of a cyclic flow and v unit chain-flows from S to T. Use the chains in this decomposition to construct a bipartite graph G having node parts S and T by inserting an edge in G joining first

and last nodes in each chain. In general, G will have
multiple edges joining the same vertices, but in any event,
G has $v = kp$ edges and the maximum valence in G is m. By
a theorem of Dulmage and Mendelsohn [1], G has a matching
consisting of $[\frac{v}{m}]$ edges that hits all nodes of valence m,
i.e. each node of valence m is incident with some edge of
the matching. (This theorem can also be derived using
flows in networks [2].) One can use this fact and induction
on k to show that G decomposes into a sum of k matchings,
each containing p edges, as follows. The case $k = 1$ is
trivial. Assume the assertion for $k - 1$ and consider k.
If $m < k$, then G has a matching M containing $p = \frac{v}{k} \leq [\frac{v}{m}]$ edges.
Moreover, the graph G-M has $kp - p = p(k-1)$ edges and
maximum valence at most k-1. If $m = k$, then G has a matching
M containing $[\frac{v}{m}] = p$ edges that hits all nodes of valence
k, and thus again G-M has $p(k-1)$ edges and maximum valence
k-1. Hence, by the induction assumption, G decomposes into
a sum of k matchings, each having p edges. It follows that
f decomposes into a sum of k matching flows, each of size p.

4. Proof of Theorem 2.2. Using the families α and β,
construct the flow network with source s and sink t shown
below:

Nodes	Directed Edges	Edge capacity function c
s, t	(s, x_i), $i \in I$	k
$S = \{x_i \mid i \in I\}$	$(x_i, y_e) \longleftrightarrow e \in A_i$	∞
$R = \{y_e \mid e \in E\}$	(y_e, y_e'), $e \in E$	1
$R' = \{y_e' \mid e \in E\}$	$(y_e', z_j) \longleftrightarrow e \in B_j$	∞
$T = \{z_j \mid j \in J\}$	(z_j, t), $j \in J$	k
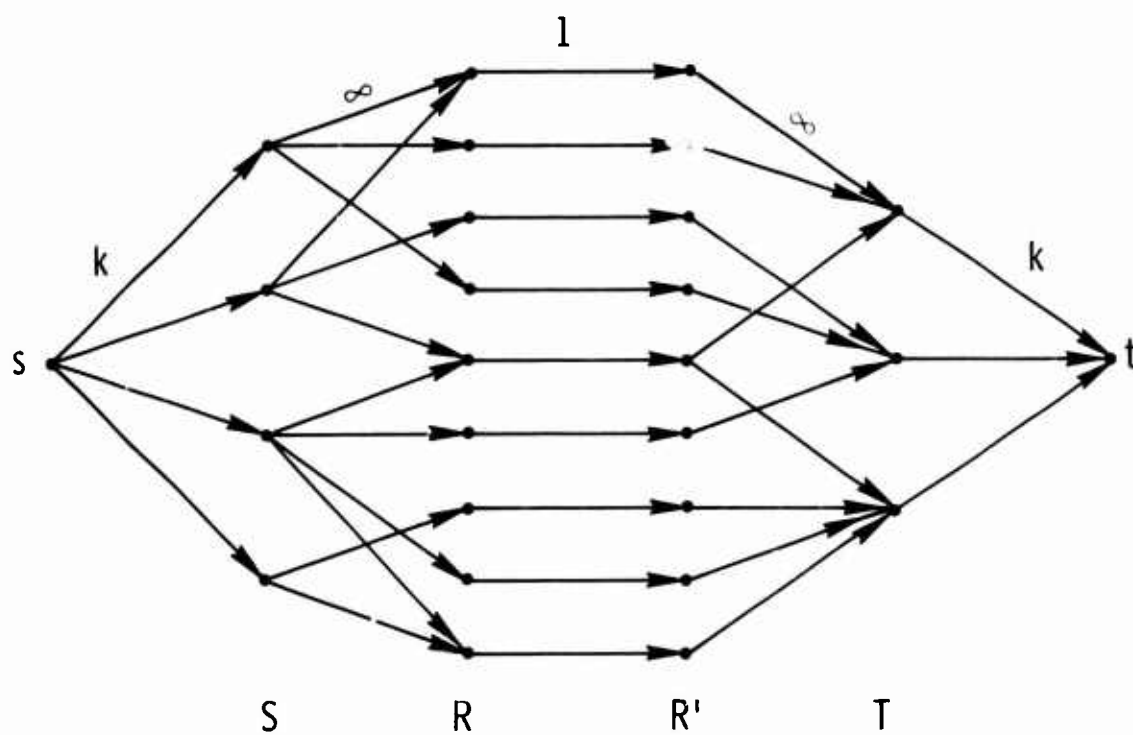
It follows from Theorem 3.1 that the families $\mathcal{A}$ and $\mathcal{B}$ have k mutually disjoint common partial transversals of size p if and only if there is an integral flow f from s to t in this network of size v = kp that satisfies the capacity constraints $f \leq c$ on all edges. By the max-flow min-cut theorem and the integrity theorem for network flows, such a flow exists if and only if all cuts separating s and t have capacities at least kp, i.e. if and only if

(4.1) $kp \leq c(X,N-X)$

for all $X \subseteq N$ such that $s \epsilon X$, $t \epsilon N-X$. Let

$$S \cap X = U \qquad S \cap (N-X) = \overline{U}$$
$$R \cap X = V \qquad R \cap (N-X) = \overline{V}$$
$$R' \cap X = V' \qquad R' \cap (N-X) = \overline{V'}$$
$$T \cap X = W \qquad T \cap (N-X) = \overline{W}$$

Then

(4.2) $c(X,N-X) = c(s,U) + c(U, \overline{V}) + c(V, \overline{V'}) + c(V',\overline{W}) +$

$$c(W,t).$$

The inequality (4.1) holds automatically unless the sets of edges $(U,\overline{V})$ and $(V', \overline{W})$ are empty. Let

$$B(\overline{V}) = \{x_i \epsilon S \mid (x_i,y_e) \text{ is an edge for some } y_e \epsilon \overline{V}\}$$
$$A(V') = \{z_j \epsilon T \mid (y'_e,z_j) \text{ is an edge for some } y'_e \epsilon V'\}.$$

Then $(U,\overline{V})$ is empty if and only if $B(\overline{V}) \subseteq \overline{U}$, and $(V',\overline{W})$ is empty if and only if $A(V') \subseteq W$. Thus (4.2) is,

if anything, decreased by taking $\bar{U} = B(\bar{V})$, $W = A(V')$.
Consequently (4.1) holds if and only if

(4.3) $kp \leq k|B(\bar{V})| + k|A(V')| + c(V,\bar{V'})$

holds for all $V \subseteq R$, $V' \subseteq R'$. Translating (4.3) into the
notation of Section 2 yields (2.4) as a necessary and
sufficient condition for the existence of k mutually disjoint
common partial transversals, each of size p, of $\mathcal{A}$ and $\mathcal{B}$.

Condition (2.3) can be derived from (4.1) in a
similar way.

A corollary of Theorem 2.2 is the following formula
for the maximum number k(p) of mutually disjoint common
partial transversals, of size p, of two families $\mathcal{A}$ and $\mathcal{B}$:

$$(4.4) \qquad k(p) = \min_{\substack{E' \subseteq E \\ E'' \subseteq E}} \left[\frac{|E-(E'\cup E'')|}{p-\#(\mathcal{A},E')-\#(\mathcal{B},E'')} \right]$$

where the minimum in (4.4) is taken over all disjoint subsets
E', E'' of E such that the denominator is positive.

Theorem 2.2 can be viewed as a generalization of the
main result in [5], which gives necessary and sufficient
conditions in order that a p by n (0,1) -matrix $A = (a_{ij})$,
with $p \leq n$, can be written as a sum

(4.5) $A = P_1 + \cdots + P_k + R,$

where each P_i has exactly one 1 in each row and at most

one 1 in each column. To see this, let $I = \{1,\cdots,p\}$,
$J = \{1,\cdots,n\}$, and define $E = \{(i,j)\epsilon\ I\times J\,|\,a_{ij} = 1\}$,
$A_i = E\cap(\{i\}\times J)$, $i\epsilon I$, $B_j = E\cap(I\times\{j\})$, $j\epsilon J$. Then a
permutation matrix P in the decomposition (4.5)
corresponds to a common partial transversal of size p of
the families $\mathcal{A} = (A_i:i\epsilon I)$ and $\mathcal{B} = (B_j:j\epsilon J)$, and Theorem 2.2
reduces to the result of [5] mentioned above.

Theorem 2.2 also gives some information on the following
maximum packing problem. Let A be the (0,1)-incidence matrix
of all common partial transversals of size p of two families
$\mathcal{A}$ and $\mathcal{B}$ of subsets of E, where the rows of A correspond to
the common partial transversals and the columns of A correspond
to members of E. Associate a weight $w(e) \geq 0$ with each
$e\epsilon E$, and ask for a solution vector y to the linear program

$$(4.6) \qquad \begin{aligned} &yA \leq w \\ &y \geq 0 \\ &\max\ 1\cdot y, \end{aligned}$$

where $1 = (1,\cdots,1)$. If $w(e) = 1$ all $e\epsilon E$, and if we
restrict y to be a (0,1)-vector in (4.6), then the formula
(4.4) is applicable. That is, k(p) in (4.4) is then equal
to max $1\cdot y$ in (4.6). The case $w(e) = 0$ or 1 can be treated
in this fashion also. Knowing how to solve the linear
program (4.6) is tantamount to knowing the blocking matrix
B for the incidence matrix A [6]. But the general maximum
packing problem (4.6) does not seem to be amenable to the
network flow approach used in this note.

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